Journal of Solid and Fluid Mechanics (JSFM), 16(3): 1-4, 2026

Shahrood University of Technology

Journal of Solid and Fluid Mechanics (JSFM)

DOI: **10.22044/jsfm.2026.17553.4051**

Manufacturing Engineering of Iran

Vibrational analysis and time response of a beam traveled by a series of moving loads

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Abstract

This paper investigates the vibrational behavior and time response of a simply supported beam traveled by a series of moving loads. The Euler-Bernoulli beam theory is employed for modeling, and the governing dynamic equations of the problem are solved analytically. Initially, the vibrational behavior of the beam caused by the passage of a single concentrated load is examined for two stages: before and after the load exits the beam. Subsequently, this analysis is extended to a series of loads moving at equal spacing, which enter the beam from the left side and exit after traversing its length. Various load movement scenarios, including motion in two opposite directions, are also studied. The effects of load velocity and the spacing between loads on the time response at the beam's midpoint and the maximum beam deflection are analyzed. The obtained results indicate that for specific combinations of load velocity and load spacing, the vibration amplitude increases due to the resonance phenomenon, which must be taken into account in beam design.

Keywords: Moving load; dynamic amplification factor; time response; Euler-Bernoulli beam; vibrational behavior.

1. Introduction

Moving loads are loads whose position changes with time and are commonly encountered in engineering structures such as bridges and railway systems. Unlike static loads, moving loads induce more complex dynamic responses, making their analysis essential for accurately predicting structural vibrations, fatigue, and resonance. The moving load problem is a classical topic in structural dynamics and has been modeled using various approaches. The simplest model considers a massless concentrated force traveling at a constant speed, while more advanced models incorporate the load's mass, stiffness, and damping to provide more realistic predictions of the structural response.

The governing equations of beams subjected to moving loads can be solved using either analytical or numerical approaches. Among the available numerical techniques, the finite element method (FEM) is one of the most widely used approaches for vibration analysis, as it enables accurate prediction of the spatial and temporal responses of beams under various moving load conditions. Owing to its versatility and accuracy, FEM has been extensively employed by many researchers in this field [1-5].

Moving loads can also be modeled using different approaches depending on the level of complexity required. In Ref. [6], the vibration response of an isotropic beam with various boundary conditions was investigated under a concentrated load moving at constant speed, accelerating, and decelerating. Ref. [7] analyzed the dynamic behavior of non-uniform multi-span beams subjected to concentrated loads traveling at both constant and variable speeds. In that study, the beam was modeled based on the Euler–Bernoulli beam theory, and the solution was obtained using modal analysis combined with the direct integration method. In another study [8], the dimensionless equation of motion of a beam subjected to a harmonically varying moving load was solved using the multiple-time-scale perturbation method. Cheung *et al.* [9] investigated the dynamic response of multi-span beams by employing modified beam vibration functions as assumed modes. They subsequently extended their work in Ref. [10] to examine the effect of moving vehicles on beam vibrations, where the vehicle was modeled as a two-degree-of-freedom system. Wong *et al.* [11] studied the nonlinear vibration of a beam under a moving load. The governing equations were derived using Hamilton's principle and solved by the Galerkin method. Their results demonstrated that both the beam geometry and material properties reduce the forced vibration amplitude while increasing the vibration frequency and the critical speed of the moving load.

Previous studies have primarily focused on the dynamic response of beams during the passage of moving loads or have assumed a constant number of loads acting on the beam. The post-passage response, particularly under multiple moving loads, has received comparatively little attention. In this study, an analytical solution is developed to investigate the beam vibration both during and after the passage of moving loads. The proposed formulation is further extended to multiple concentrated loads moving in the same or opposite directions and is used to evaluate the effects of load velocity, load spacing, opposing load motion, and damping ratio on the beam's dynamic response, maximum deflection, and resonance behavior.

2. Problem Formulation

The governing equation for the transverse vibration of a beam subjected to a moving concentrated load can be expressed as follows [12]:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = \delta(x - vt)P \quad (1)$$

where EI is the flexural rigidity of the beam, while ρ and A denote the material density and cross-sectional area of the beam, respectively. P is the magnitude of the moving load, which travels along the beam at a constant velocity v, and $\delta(x)$ represents the Dirac delta function. The solution of (1) based on separation of variables is defined as:

$$w(x,t) = \sum_{n=1}^{\infty} y_n(t)\sin(\frac{n\pi x}{L}) \quad (2)$$

Where $y_n(t)$ represents the nth modal coordinate and $\sin(n\pi x/L)$ is the nth mode shape of a simply supported beam. by substituting (2) in (1), the time response of beam is obtained from the following differential equation:

$$\ddot{y}_n(t) + \omega_n^2 y_n(t) = \left(\frac{2P}{\rho AL}\right) sin(\Omega_n t) \quad (3)$$

Where ω_n and Ω_n are the natural frequency of the beam and the excitation frequency of moving load, respectively.

The solution of (3) is represented as:

$$y_n(t) = \left(\frac{2P}{\rho AL\omega_n^2}\right)\left(\frac{1}{1-\beta_n^2}\right)[\sin(\Omega_n t) - \beta_n \sin(\omega_n t)], \;\; \beta_n \neq 1 \quad (4\text{-}1)$$

$$y_n(t) = \left(\frac{2P}{\rho AL\omega_n^2}\right)\left(\frac{1}{2}\right)[\sin(\omega_n t) - \omega_n t \sin(\omega_n t)], \;\; \beta_n = 1 \quad (4\text{-}2)$$

Where $\beta_n = \Omega_n/\omega_n$ is the frequency ratio. For $t > \tau$, after the moving load has completely left the beam, the beam undergoes free vibration, and its dynamic behavior is governed by the corresponding free vibration problem. The initial conditions of this free vibration problem at time $t = \tau$ can be defined as:

$$w(x,0) = y_n(\tau)\sin\left(\frac{n\pi x}{L}\right)$$
$$\dot{w}(x,0) = \dot{y}_n(\tau)\sin\left(\frac{n\pi x}{L}\right) \quad (5)$$

The time-dependent response can therefore be expressed as (6), where $t_1 = t - \tau$ denotes the elapsed time after the moving load has completely left the beam:

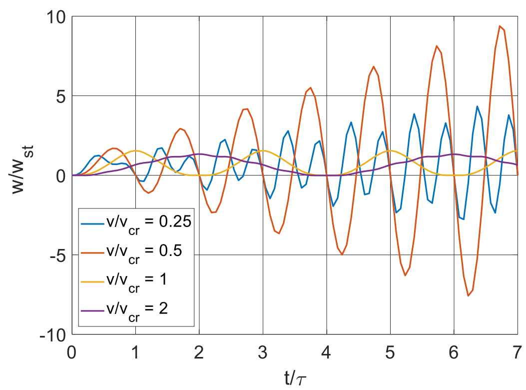
$$y_n(t_1) = y_n(\tau)cos(\omega_n(t_1)) + \frac{\dot{y}_n(\tau)}{\omega_n} sin(\omega_n(t_1)) \quad (6)$$

This solution procedure can be readily extended to the case of multiple moving loads traveling at the same velocity and with equal spacing. Owing to the linearity of the governing differential equation, the overall response can be obtained by applying the principle of superposition. To analyze the vibration behavior of a beam subjected to two groups of moving loads entering the beam from opposite ends and leaving it after traversing the beam length, the solution given in (2) is modified as follows:

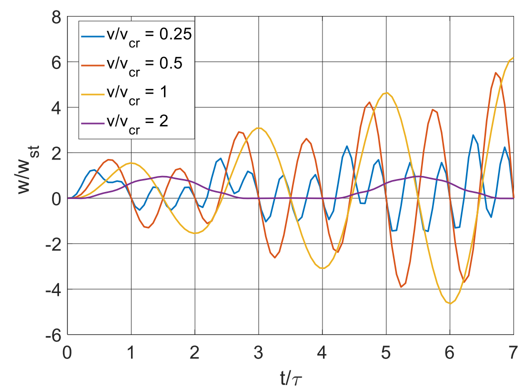
$$w(x,t) = \sum_{n=1}^{\infty} y_n(t)\left(\sin\left(\frac{n\pi x}{L}\right) + \sin(\frac{n\pi(L-x)}{L})\right) \quad (7)$$

3. Results and Discussion

Figure 1 illustrates the time response of the beam midpoint subjected to a series of moving loads traveling at the same velocity and with equal spacing. The midspan displacement is plotted as a function of time for different load velocities and load spacings. As can be observed, for certain combinations of load velocity and spacing, the vibration amplitude increases progressively with time. In the absence of damping, this leads to the occurrence of resonance, which may result in excessive structural vibrations and potentially cause serious damage to the structure. To provide a clearer understanding of this phenomenon, Figure 2 presents the dynamic amplification factor as a function of the load speed ratio for different spacings between consecutive moving loads. The dynamic magnification factor is defined as: $D_d = w_{max}(\frac{L}{2},t)/w_{st}(\frac{L}{2})$.



d/L=1



d/L=2

Figure 1. Time histories of the beam midspan vibration under a series of moving loads traveling in the same direction for different load speed ratios and load spacings.

A detailed examination of the response curves reveals the following relation between load distance ratio (d/L) and speed ratio (v/v_{cr}), which causes peak points or resonance in the vibration of beam's midspan:

$$\frac{d}{L} = 2n\frac{v}{v_{cr}}, n = 1,2,3, \dots \tag{8}$$

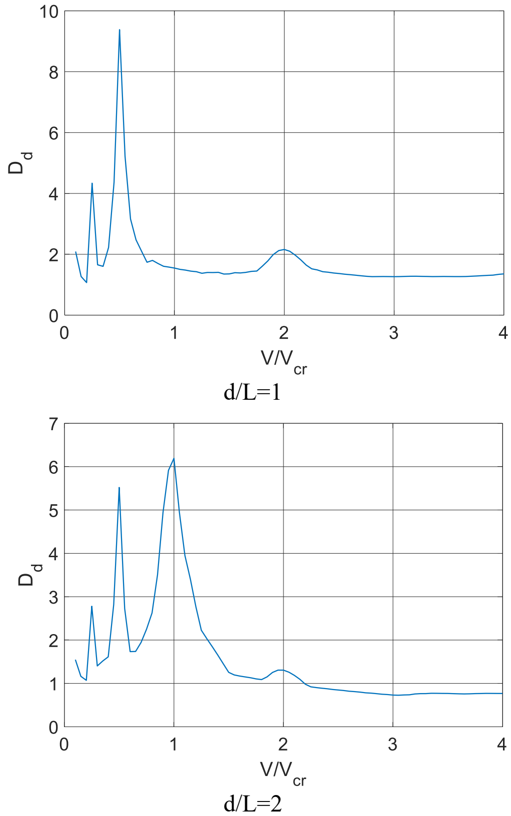


Figure 2. Dynamic amplification factor versus load speed ratio for a beam subjected to a series of moving loads traveling in the same direction.

Next, a similar analysis was carried out for the case in which two groups of moving loads enter the beam simultaneously from opposite ends and leave the beam after traversing its length. The results exhibit trends similar to those observed for the case of loads traveling in the same direction (from the left to the right end of the beam). However, in the present case, both the vibration amplitude and the dynamic amplification factor are significantly greater.

As discussed earlier, structural damping reduces the vibration amplitude, particularly under resonance conditions. To evaluate the sensitivity of the system to the structural damping ratio and its influence on the vibration response of a beam subjected to moving loads, Figure 3 presents the dynamic amplification factor for a series of moving loads traveling in the same direction for different damping ratios. As shown, increasing the damping ratio significantly reduces the maximum vibration amplitude, while the locations of the resonance peaks remain unchanged. Consequently, relation (8) remains valid even in the presence of damping. Furthermore, Figure 4 illustrates the time history of the beam midspan vibration for one of the resonance cases by considering structural damping. The results clearly demonstrate that the presence of damping, and particularly an increase in the damping ratio, effectively suppresses the vibration amplitude.

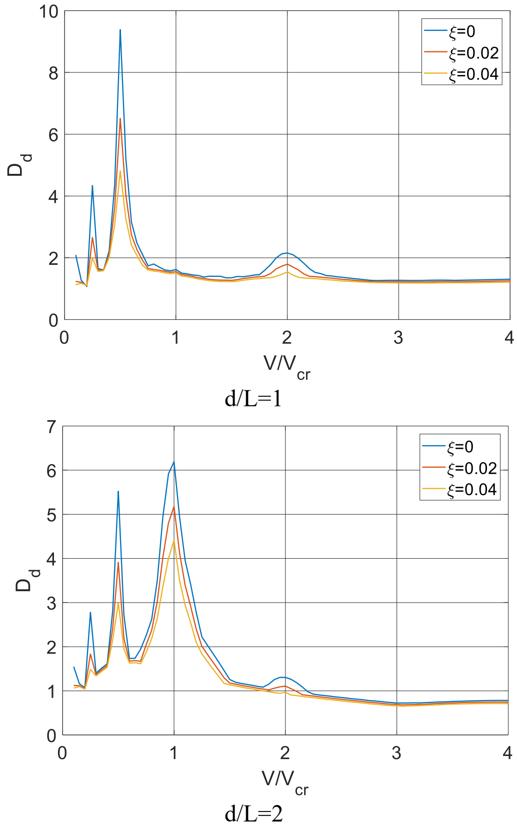


Figure 3. Effect of the damping ratio on the dynamic amplification factor of a beam subjected to a series of moving loads traveling in the same direction.

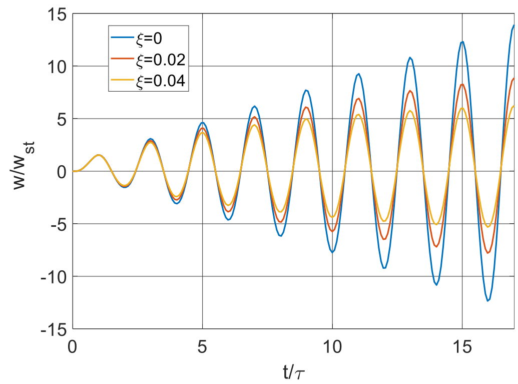


Figure 4. Comparison of the time histories of the beam midspan vibration for different damping ratios under the resonance condition $v/v_{cr} = 1$ and $d/L = 2$.

4. Conclusions

This study presents an analytical investigation of the vibration response of a simply supported Euler–Bernoulli beam subjected to multiple moving concentrated loads traveling with equal velocities and spacing in the same or opposite directions. The analytical solution is developed for both the load-passage and post-passage stages, with particular emphasis on the beam response after the moving loads leave the beam. The formulation is first established for a single moving load and then extended to multiple moving loads. The main conclusions of this study are summarized as follows:

1. The antisymmetric vibration modes are not excited by the moving loads and therefore do not affect the beam response. Moreover, higher load velocities lead to faster solution convergence.
2. For multiple moving loads, resonance occurs at specific combinations of load speed and spacing, resulting in a significant increase in vibration amplitude. When loads travel in opposite directions, the response follows a similar trend but with substantially larger vibration amplitudes.
3. Structural damping significantly reduces the vibration amplitude, especially under resonance conditions, without changing the resonance locations.

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