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Solving the Forward Kinematics of Planar Parallel Mechanisms Based on Interval Analysis and Mean-Value Form

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Abstract

In this paper, a new method is presented for analyzing the forward kinematics problem of planar parallel mechanisms based on a combination of interval analysis and a refinement process. In the first step, the kinematic equations of each chain of the parallel mechanism are derived by applying the relevant kinematic constraints. Then, the forward kinematics problem is solved with the desired accuracy using a combination of interval analysis and mean-value form. For validation purposes, the results of the proposed method are compared with those obtained from the resultant method. The implementation of the proposed approach on a 3-RRR planar parallel mechanism demonstrates its capability to accurately estimate the position and orientation of the moving platform. The main advantage of this method is its high accuracy with reasonable computational efficiency, which makes it a promising option for solving the forward kinematics problem of planar parallel mechanisms.

Keywords: Forward kinematics; Planar parallel mechanism; Interval analysis; Mean-value form; Resultant method.

1. Introduction

The forward kinematics problem of parallel mechanisms concerns determining the pose (position and orientation) of the moving platform from given values of the actuated joint variables (for example, lengths of prismatic actuators or the driving joint angles). Analysis of forward kinematics typically employs methods such as algebraic geometry [1], variable elimination [2], and intelligent algorithms based on machine learning and artificial neural networks [3].

Interval analysis is a powerful computational mathematics tool for solving nonlinear equations and optimization problems [4]. The INTLAB toolbox is a MATLAB-based environment developed to perform interval computations and to analyze their results. Several studies have investigated the use of interval analysis in parallel mechanisms, focusing mainly on forward kinematics [5], sensitivity analysis [6], and workspace analysis [7].

Developing efficient methods that can obtain unique and robust solutions remains an open challenge in the field of parallel mechanisms. In this work, a hybrid approach is proposed that combines interval analysis with refinement techniques. Interval analysis provides a rigorous mathematical framework that enables computations with guaranteed reliability; the mean-value form-based refinement systematically improves result accuracy. The fusion of these two approaches yields an efficient computational framework for solving the forward kinematics of planar parallel mechanisms that both increases numerical accuracy and enables consideration of all possible configuration branches.

2. Proposed algorithm for forward kinematics analysis of planar parallel mechanisms

To perform forward kinematic analysis using interval techniques, first the forward kinematics equation associated with each kinematic chain of the parallel mechanism is derived in Euclidean three-dimensional space. Given the forward kinematics problem of a parallel mechanism, the actuated joint values are provided as fixed real values to the system of nonlinear equations. Consequently, a system of three nonlinear equations is obtained, denoted by f_1, f_2 f_3, with three unknowns x, y, θ corresponding to the platform pose.

Next, initial intervals are assigned to each unknown and placed into the system of equations. The output interval of each equation is calculated by applying interval arithmetic and performing refinement based on the mean-value form. The necessary condition for the existence of a solution in the interval system is that the interval of each equation contains zero — i.e., $0 \in f_i$, $i = 1,2,3$. Then the largest interval among x, y, θ is identified and bisected into two subintervals. Each subinterval is substituted into the system separately, and the interval ranges f_i is recomputed, and the zeroinclusion condition is rechecked. Only those subintervals for which all interval equations still contain zero are retained; the others are discarded The process of dividing the intervals and checking for the existence of a solution continues until the width of the intervals for x, y, θ reach the prescribed tolerance ε. Finally, the remaining intervals collectively encompass all forward kinematic solutions of the parallel mechanism with the specified accuracy.

3. Results: Solving the forward kinematics of the 3-RRR parallel mechanism using the proposed algorithm

In this section, the proposed algorithm is used to analyze the forward kinematics problem of a 3-DOF 3-RRR planar parallel mechanism. The general schematic of this mechanism is shown in Fig. 1. The forward kinematic equation of the i^{th} kinematic chain, f_i, is obtained for input values $A_1 = [0, 0]$, $A_2 = [a, 0]$, $A_3 = [\frac{a}{2}, \frac{a\sqrt{3}}{2}]$ and the design parameters of Table 1 by writing the equation B_iC_i and substituting the values of B_i and C_i.

After deriving the forward kinematics equations for each kinematic chain of the 3-RRR parallel mechanism, the actuated joint angles (θ_1, θ_2, θ_3), the maximum number of iterations (N_{max}) and the error value (ε) are entered into the algorithm. The error value is also considered to be $\varepsilon = 0.0001$. The intervals $x_p = [-10\ 150]$, $y_p = [-20\ 140]$, $\theta_p = [0^\circ\ 360^\circ]$ are chosen as initial intervals to cover the entire accessible search space. Three interval equations are obtained for each iteration by substituting the intervals into the kinematic equations.

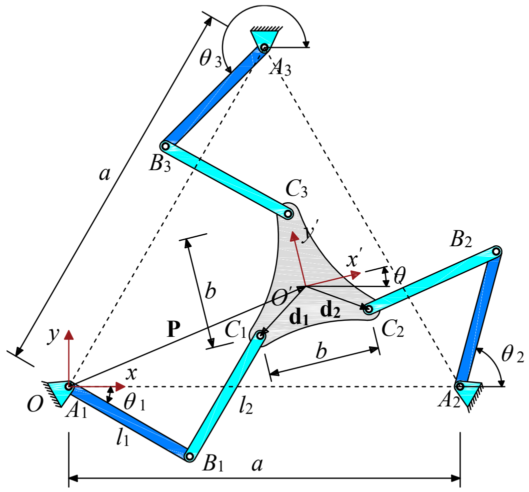

Figure 1- Schematic representation of the 3-RRR parallel mechanism

Table 1- Design parameters of the 3-RRR parallel mechanism

a (cm)	b (cm)	l_1 (cm)	l_2 (cm)
140	40	50	50

Subsequently, the intervals of functions f_1, f_2, f_3 are calculated at each iteration by applying the rules of interval arithmetic and performing refinement using the mean-value form. If the value zero is presented in all obtained intervals for the kinematic equations ($0 \in f_1, 0 \in f_2$ and $0 \in f_3$), the interval is divided into two subintervals, and this process is repeated on them. The algorithm iterates until the width of the intervals reaches the desired accuracy (ε). Finally, the solutions of the forward kinematics problem of the parallel mechanism are obtained in the form of intervals with the specified accuracy.

For example, the results of the proposed algorithm on the 3-RRR planar parallel mechanism are shown in Figs 2-a to 2-d for different joint angle values. It should be noted that due to the limitations of two-dimensional plotting, only the intervals for x and y can be graphically represented. However, the results of three output intervals of the algorithm (x_p, y_p, θ_p) are provided in Table 2 corresponding to the aforementioned figures. It is observed that the analysis of the forward kinematics problem using the proposed algorithm resulted in six intervals for the joint angles $\theta_1 = 60^\circ$, $\theta_2 = 150^\circ$, $\theta_3 = 240^\circ$ and also for $\theta_1 = 45^\circ$, $\theta_2 = 120^\circ$, $\theta_3 = 270^\circ$, four intervals for $\theta_1 = -30^\circ$ ،$\theta_2 = 180^\circ$, $\theta_3 = 270^\circ$, and two intervals for For $\theta_1 = 90^\circ$, $\theta_2 = 120^\circ$, $\theta_3 = 300^\circ$. The lower and upper bounds of all remaining intervals are the same up to three decimal places, as the error is assumed to be $\varepsilon = 0.0001$. Therefore, each interval can be considered equivalent to a real number with three decimal places of accuracy.

The resultant method is used to evaluate the results of the proposed algorithm, and the results of this method are compared with the interval results obtained from the

Table 2- Intervals of the forward kinematics problem of the 3-RRR parallel mechanism for different joint angles and $\varepsilon = 0.0001$

$\theta_1 = 60^\circ$ $\theta_2 = 150^\circ$ $\theta_3 = 240^\circ$	$x = [48.788\ 48.788]$ $y = [2.572\ 2.572]$ $\theta = [-11.07^\circ\ -11.07^\circ]$	$x = [24.495\ 24.495]$ $y = [15.664\ 15.664]$ $\theta = [25.90^\circ\ 25.90^\circ]$
	$x = [65.474\ 65.474]$ $y = [63.176\ 63.176]$ $\theta = [-94.60^\circ\ -94.60^\circ]$	$x = [52.281\ 52.281]$ $y = [37.451\ 37.451]$ $\theta = [102.71^\circ\ 102.71^\circ]$
	$x = [65.101\ 65.101]$ $y = [91.954\ 91.954]$ $\theta = [-30.55^\circ\ -30.55^\circ]$	$x = [86.280\ 86.280]$ $y = [84.301\ 84.301]$ $\theta = [7.61^\circ\ 7.61^\circ]$
$\theta_1 = 45^\circ$ $\theta_2 = 120^\circ$ $\theta_3 = 270^\circ$	$x = [98.711\ 98.711]$ $y = [12.169\ 12.169]$ $\theta = [-17.95^\circ\ -17.95^\circ]$	$x = [55.154\ 55.154]$ $y = [8.573\ 8.573]$ $\theta = [33.29^\circ\ 33.29^\circ]$
	$x = [83.748\ 83.748]$ $y = [45.554\ 45.554]$ $\theta = [-99.07^\circ\ -99.07^\circ]$	$x = [64.894\ 64.894]$ $y = [41.097\ 41.097]$ $\theta = [111.96^\circ\ 111.96^\circ]$
	$x = [61.856\ 61.856]$ $y = [94.847\ 94.847]$ $\theta = [-9.90^\circ\ -9.90^\circ]$	$x = [77.310\ 77.310]$ $y = [96.357\ 96.357]$ $\theta = [21.40^\circ\ 21.40^\circ]$
$\theta_1 = -30^\circ$ $\theta_2 = 180^\circ$ $\theta_3 = 270^\circ$	$x = [45.140\ 45.140]$ $y = [9.314\ 9.314]$ $\theta = [-63.40^\circ\ -63.40^\circ]$	$x = [67.182\ 67.182]$ $y = [29.784\ 29.784]$ $\theta = [87.72^\circ\ 87.72^\circ]$
	$x = [26.094\ 26.094]$ $y = [10.578\ 10.578]$ $\theta = [-18.84^\circ\ -18.84^\circ]$	$x = [32.144\ 32.144]$ $y = [39.918\ 39.918]$ $\theta = [25.80^\circ\ 25.80^\circ]$
$\theta_1 = 90^\circ$ $\theta_2 = 120^\circ$ $\theta_3 = 300$	$x = [59.545\ 59.545]$ $y = [92.162\ 92.162]$ $\theta = [-12.54^\circ\ -12.54^\circ]$	$x = [52.702\ 52.702]$ $y = [27.493\ 27.493]$ $\theta = [3.56^\circ\ 3.56^\circ]$

proposed algorithm. The resultant elimination method is an effective way to progressively reduce a system of equations into a univariate expression. In this method, one or more variables are eliminated from the system of equations. The degree of the final univariate expression is equal to the number of possible solutions for the system of equations. To find all solutions, the roots of the final univariate expression must be substituted back into the previous equations to extract the common solutions.

The results of the forward kinematics analysis using the resultant method show that the final univariate expression for the 3-RRR parallel mechanism is always of the sixth degree for any set of joint angles $\theta_1, \theta_2, \theta_3$. This indicates that, in general, the 3-RRR parallel mechanism have a maximum of six solutions for the forward kinematics problem. In cases where all roots of the final univariate expression are real, six poses (positions and orientations) can be obtained for the moving platform. However, if some roots are complex, only the real roots are considered as physical solutions. Finally, all solutions of the forward kinematics problem are extracted by substituting the real roots of the final univariate expression into the preceding equations. A review of the results for various joint angles shows that the solutions obtained from the proposed interval algorithm are in complete agreement with the results obtained from the resultant method. This also demonstrates the high accuracy of the proposed algorithm in solving the forward kinematics problem of the parallel mechanism.

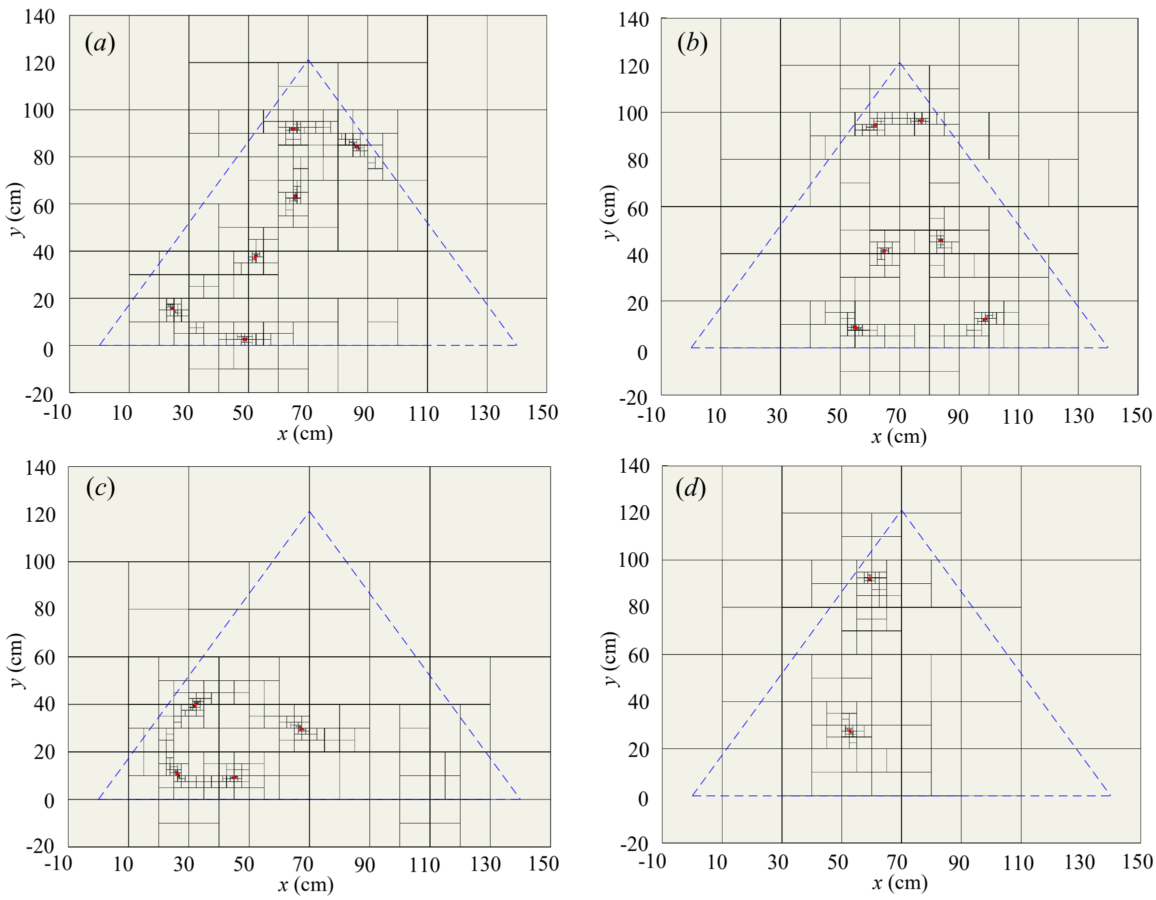

Figure 2- Analysis of the forward kinematics problem of the 3-RRR parallel mechanism for the joint angles

***a)* θ_1=60°, θ_2=150° and θ_3=240°, *b)* θ_1=45°, θ_2=120° and θ_3=270°**

***c)* $\theta1$=-30°, $\theta2$=180° and $\theta3$=270° *d)* θ_1=90°, θ_2=120° and θ_3=300°**

4. Conclusions

In this paper, a hybrid approach was proposed based on interval analysis and mean-value form to solve the forward kinematics problem of planar parallel mechanisms. The evaluation of this method on a 3-RRR parallel mechanism and its comparison with the resultant method demonstrated that the proposed approach enables the comprehensive identification of all possible solutions within the search space by leveraging the capabilities of interval analysis. The accuracy of the results was systematically improved by employing the refinement algorithm and reducing the excess width of the intervals. Other advantages of this method include guaranteeing full coverage of valid solutions, stability against rounding errors, high computational accuracy, and comprehensive coverage of the possible workspace of the parallel mechanism.

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