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Tissue Temperature Control in a Bio-Heat Transfer Equation with the Conjugate Gradient Method

F. Davoodi[1], H. Ahsani Tehrani[2], M. Nazri[3]

[1]Ph.D. Student, Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran
[2]Assoc. Prof., Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran
[3]Assoc. Prof., Faculty of Mechanical Enginerring, Shahrood University of Technology, Shahrood, Iran

*Corresponding author: hahsani@shahroodut.ac.ir

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Abstract

The classical conduction heat transfer model which considers infinite thermal propagation speed, named Fourier model and its equations are in elliptic form and has numerous applications. This model is not appropriate for many industrial applications, especially in medical applications and thus hyperbolic or non-Fourier model that considers finite heat propagation speed should be used. Temperature control in certain points of such systems obeying these types of equations is an important problem which has been studied in this paper. In this paper, first, the validity of the conjugate gradient method is approved using a known heat flux at a system boundary and then the method is used to estimate the boundary condition which leads to a desired temperature distribution in the geometry. Additionally, the modeling and inverse problem solution are studied for noise in input data and results showed appropriate accuracy and convergence even for considerable noise in input data.

Keywords: Hyperbolic Equation, Non-Fourier Heat Transfer, Inverse Problem, Conjugate Gradient Method.

1. Introduction

With the advancement of technology in various fields such as industry and medicine, and the entry of technology into small-scale areas, both in terms of time and space, the classical theory of Fourier heat conduction, in which the speed of heat propagation is infinite, provides unacceptable results. Therefore, it is necessary to use new heat transfer models based on limited heat propagation speed. Previously, heat transfer processes have been extensively studied using the classical Fourier heat conduction model and the hyperbolic Fourier model. This model assumes an infinite heat propagation speed, which is physically impossible. The hyperbolic heat conduction model has a more realistic physics by considering limited heat propagation speed for heat propagation.

However, solving such equations mathematically is challenging. Nevertheless, recent research has been conducted to improve proposed models and develop solutions for these problems. Since solving these equations may face numerous mathematical problems (such as thermal discontinuity), various numerical methods have been proposed and developed to address these challenges. Heat transfer problems can be classified into two categories: direct problems and inverse problems. The aim of solving direct problems is to obtain the temperature distribution given the initial conditions, heat flux, and thermodynamic properties. In contrast, the aim of solving inverse problems is often to obtain the unknown heat flux based on measured or desired temperature. In direct heat transfer problems, the cause (heat flux at the boundary) is known, and the effect (temperature field inside the body) is determined by solving governing equations. In contrast, inverse heat transfer problems involve estimating the unknown cause based on the known or desired effect. The inverse heat transfer method has various applications in different fields of science and engineering. However, the use of the conjugate gradient method in solving inverse problems has a limited history. Lin and Chen [1] estimated the boundary heat flux in heat transfer problems using a similar method. Park and Chung [2] compared the direct differentiation method with the use of an adjoint equation to calculate gradients in the conjugate-gradient method. Their results showed that although the ordinary differentiation method is simple and accurate, it increases computation time. The use of an adjoint equation to calculate gradients reduces computation time significantly. Jarny et al. [3] investigated a multi-dimensional inverse conductive heat transfer problem using the conjugate gradient method and the adjoint equation. Kowsary et al. [4,5] also investigated inverse problems involving heat conduction and radiation using both the conjugate-gradient and variable metric methods.

Mohammadiun and Molavi also studied inverse conduction heat transfer problems using the conjugate gradient method [6]. Additionally, Mahdavi and Khalkhalian examined the estimation of heat flux on the cylinder wall in an internal combustion engine [7], as well as convective heat flux through the nozzle throat of a rocket engine [8]. Lee et al. employed the conjugate gradient method to estimate heat flux in living tissue [9]. Goudarzi and Azimi studied direct bio-heat transfer with dual-phase lag [10], while Alosaimi and Lesnic estimated location-dependent heat sources for hyperbolic bio-heat transfer using the conjugate gradient method, demonstrating good accuracy even in two-dimensional cases [11].The effect of three phase lag in bio-heat transfer equation is studied recently by Kumar et al. [12]

Given the numerous capabilities of the conjugate gradient method for solving many inverse problems, this method has been used in this article to estimate heat flux in non-Fourier heat transfer problems. The aim of this study is to investigate the ability of the conjugate gradient method in solving inverse heat transfer problems in the form of hyperbolic equations (non-Fourier) and temperature control in these problems. This method transforms the inverse problem into three sub-problems: direct, sensitivity, and adjoint, that will be discussed later.

2. Governing Equations

2.1. Direct Problem

The governing equations of the direct problem for the 1D geometry with the known boundary conditions are as follows:

$$k\frac{\partial^2 T}{\partial x^2} = \tau\rho c\frac{\partial^2 T}{\partial t^2}+\rho c\frac{\partial T}{\partial t};0<x<l,t>0 \quad (1\text{-}1)$$

$$T(x,0)=T_0 \qquad 0<x<l \quad ,t=0 \quad (1\text{-}2)$$

$$\frac{\partial T(x,0)}{\partial t}=0 \;; \qquad 0<x<l \quad ,t=0 \quad (1\text{-}3)$$

$$\frac{\partial T(0,t)}{\partial x}=0 \;; \qquad x=0 \quad (1\text{-}4)$$

$$k\frac{\partial T(l,t)}{\partial x}=q(l,t)+\tau\frac{\partial q(l,t)}{\partial t} \;;x=l,t>0 \quad (1\text{-}5)$$

Here, c, ρ, k, and τ represent thermal conductivity, density, specific heat capacity, and relaxation time respectively. If all boundary conditions are known, the problem is a direct problem and can be solved using the implicit method (Crank-Nicolson).

2.2. Inverse Problem

To solve the mentioned inverse equations, the heat flux at one of the boundaries is assumed to be unknown, while the other equations (1-1) to (1-5) are given. Additionally, this inverse problem requires an extra known condition which is the temperature at a specific location. The temperature measured by a sensor at that location is indicated by Y (0, T), can also be used. The desired or measured temperature may include measurement errors that will be artificially applied to the data. Therefore, the inverse problem can be defined as "estimating the unknown boundary heat flux using measured or known temperature." Solving this inverse problem involves minimizing the following objective function.

$$J[q(l,t)]=\int_{t=0}^{t_f}[T(0,t)-Y(0,t)]^2 dt \quad (2)$$

2.3. Sensitivity problem

The sensitivity problem can be derived by perturbing the heat flux and temperature by a small amount, substituting in the direct problem and deducing from the equation (1-1) to (1-5) as follows.

$$k\frac{\partial^2 \Delta T}{\partial x^2} = \tau\rho c\frac{\partial^2 \Delta T}{\partial t^2}+\rho c\frac{\partial \Delta T}{\partial t};0<x<l,t>0 \quad (3\text{-}1)$$

$$\Delta T(x,0)=0,0<x<l,t=0 \quad (3\text{-}2)$$

$$\frac{\partial \Delta T(x,0)}{\partial t}=0 \qquad ,0<x<l,t=0 \quad (3\text{-}3)$$

$$\frac{\partial \Delta T((0,t)}{\partial x}=0 \qquad ,x=0 \qquad ,t>0 \quad (3\text{-}4)$$

$$\frac{k\partial \Delta T(l,t)}{\partial x} = \Delta q(l,t)+\tau\frac{\partial \Delta q(l,t)}{\partial t},x=l,t>0 \quad (3\text{-}5)$$

2.4. Adjoint problem and gradient equation

Multiplying the direct equations by Lagrange coefficients, integrating over the spatial and temporal domains and after some mathematical progress leads to the following adjoint problem.

$$k\frac{\partial^2 \lambda}{\partial x^2}=\tau\rho c\frac{\partial^2 \lambda}{\partial t^2}+\rho c\frac{\partial \lambda}{\partial t} \;;0<x<l,\,t>0 \quad (4\text{-}1)$$

$$\lambda(x,t_f)=0 \qquad 0<x<l \quad t=t_f \quad (4\text{-}2)$$

$$\frac{\partial \lambda(x,t_f)}{\partial t}=0 \qquad 0<x<l \quad t=t_f \quad (4\text{-}3)$$

$$\frac{\partial \lambda(0,t)}{\partial x}=\frac{-2(T-Y)}{k} \quad x=0 \quad t>0 \quad (4\text{-}4)$$

$$\frac{\partial \lambda(l,t)}{\partial x}=0 \qquad x=l \qquad t>0 \quad (4\text{-}5)$$

The objective function gradient can be computed using the solution of the adjoint problem.

$$J'[q(l,t)] = \lambda(l,t)-\tau\lambda(l,t)\delta(l,t)\delta(t-0)-\tau\frac{\partial \lambda(l,t)}{\partial t} \quad (5)$$

2.5. Conjugate gradient method

The following iterative conjugate gradient method is used to minimize the objective function:

1- Solve the direct problem with the initial guess.

2- Check the stopping criteria and continue the process when it is not satisfied.

3- Solve the adjoint problem and compute the gradient of the objective function.

4- Evaluate the conjugate coefficient and direction of descent using the following equations:

$$\gamma^{(n)}=\frac{\int_{t=0}^{t_f}\left[J'^{(n)}\right]^2 dt}{\int_{t=0}^{t_f}\left[J'^{(n-1)}\right]^2 dt} \quad , \quad \gamma^{(0)}=0$$

$$P^{(n)}(l,t)=J'^{(n)}(l,t)+\gamma^{(n)}P^{(n-1)}(l,t)$$

5- Solve the sensitivity problem by putting $\Delta q=P^{(n)}$

6- Compute the search step size as follows:

$$\beta^{(n)}=\frac{\int_{t=0}^{t_f}[T(0,t)-Y(0,t)]\Delta T(0,t)dt}{\int_{t=0}^{t_f}[\Delta T(0,t)^2]dt}$$

7- Update the heat flux with the following equation and continue.

$$q^{(n+1)}(l,t)=q^{(n)}(l,t)-\beta^{(n)}P^{(n)}(l,t)$$

3. Results and Discussion

After validation of the numerical method for solving the direct problem, three different shapes desired temperatures are used to evaluate the performance of the proposed method including sinusoidal, triangle and step shape functions. The results showed a very good correlation between desired and computed temperatures even with the noisy input data. The desired and estimated temperatures for aforementioned functions are illustrated in Figs. (1) to (3).

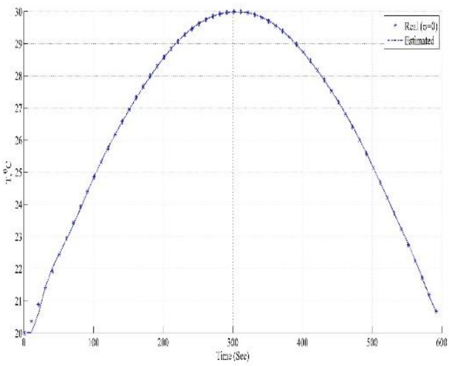

Figure 1. Desired and estimated sinusoidal shape temperature

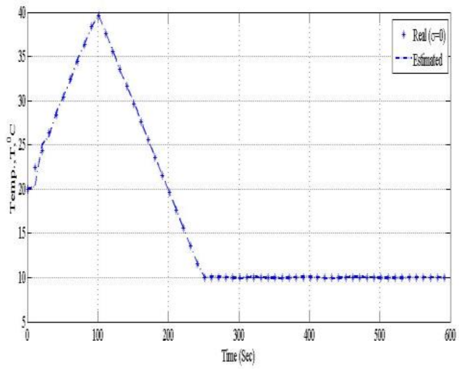

Figure 2. Desired and estimated triangular shape temperature

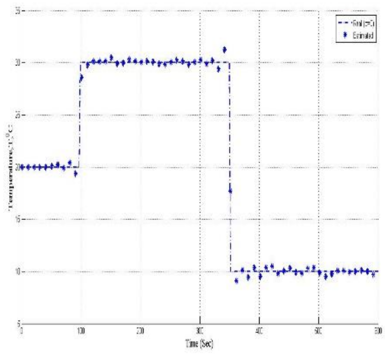

Figure 3. Desired and estimated step shape temperature

4. Conclusions

In this article, the conjugate gradient method was used to estimate the transient boundary heat flux in inverse heat conduction problems, aiming to control the temperature at specific points within the computational domain. Two examples with distinct desired temperature profiles were considered to evaluate the accuracy and validity of the obtained results. The results demonstrate the capability of the proposed method, even when dealing with disturbances in the input data.

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