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Guided wave modeling in a plate with infinite width using p-version of semi-analytical finite element method

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Abstract

Balancing accuracy and computational cost in modeling and analyzing engineering problems has always been a crucial concern. The same principle applies to the modeling of wave propagation in structures and deriving dispersion curves, which are critical in structural health monitoring and material property identification. The importance of maintaining this balance appears especially in cases where there is a need for repetition in modeling. In this study, we aimed to improve the accuracy and computational cost of the semi-analytical finite element method by incorporating hierarchical shape functions. The results indicate that using appropriate shape functions can enhance the performance of the semi-analytical finite element method for modeling wave propagation in structures. The study also investigated the impact of the number of degrees of freedom on the calculation of the cutoff frequency, the accuracy of the dispersion curves, and the increase in modeling error resulting from this factor.

Keywords: SAFE, dispersion relation, guided wave, cut-off frequency

1. Introduction

The semi-analytical finite element method (SAFE) was introduced in 1973 by Alami [1] and Lagas [2] for modelling wave propagation in structures with arbitrary cross section. This method, which has been used by others in civil engineering [3], is closely related to the strip element method and the thin element method [4]. In this method, the cross section of the structure is discretized by elements and an analytical expression is considered in the direction of wave propagation. This method has been further developed over the years by other researchers to include structures with more complex geometries [5-7] and other materials [8, 9].

An additional factor that has been noticed is the use of different shape functions in the semi-analytical finite element method. The different methods used to discretize the cross section of the structure can be divided into one-dimensional (1D SAFE) and two-dimensional (2D SAFE) semi-analytical methods. For structures with simple geometry, such as plates and cylindrical shells, the 1D SAFE method can be used to reduce the calculations. In this method, one-dimensional three-noded elements with second-order shape functions and three degrees of freedom for displacement in each node have been commonly considered [10-15], while two-noded elements [16] and three nodes with two degrees of freedom in each node [6] have also been used by researchers. In the 2D SAFE method, triangular elements with three nodes and quadrilateral elements with four nodes are most commonly used [17-22]. In some cases, triangular elements with six nodes, quadrilateral elements with eight nodes [23-27], and rarely quadrilateral elements with nine nodes [25] have also been proposed by researchers. Despite numerous researches on the semi-analytical finite element method using low-order shape functions, little research has been done on the use of high-order shape functions.

According to the mentioned investigations, the innovation in this article lies in the combination of hierarchical shape functions based on the normalized integral of Legendre polynomials with the SAFE method and the presentation of the p-version of semi-analytical finite element method. This article is organized as follows. In part 2, SAFE method formulation is presented. Hierarchical shape functions are studied in Section 3. The numerical modeling and discussion are presented in Section 4, and finally the conclusions are drawn in Section 5.

2. Semi-analytical finite element method

To use the 1D SAFE method to model wave propagation in a plate with infinite width or a pipe, at each node in a one-dimensional element, three degrees of freedom are considered and furthermore an analytical term $e^{-i(\omega t-kx)}$ is considered. In the considered formulation, k stands for the wave number, x for the direction of wave propagation, ω for the angular frequency and t for the time. Using this method, the displacement relations in the element are presented as follows [6]:

$$\boldsymbol{u}^{(e)}(x,z,t) = \begin{bmatrix} u_x^{(e)} \\ u_y^{(e)} \\ u_z^{(e)} \end{bmatrix} = \boldsymbol{N}(z)\boldsymbol{q}^{(e)} e^{-i(\omega t - kx)} \quad (1)$$

In the above relation, $\boldsymbol{N}$ represents the shape function and $\boldsymbol{q}^{(e)}$ represents the displacement vector of the nodes in each element. By calculating the kinetic energy, ϕ, and the potential energy, K, and placing them in the Hamiltonian relation, and after performing the relevant calculations, the equation of motion for an element can be derived [6]:

$$\left(\boldsymbol{k}_1^{(e)} + i\kappa \boldsymbol{k}_2^{(e)} + \kappa^2 \boldsymbol{k}_3^{(e)} - \omega^2 \boldsymbol{m}^{(e)}\right) \boldsymbol{Q}^{(e)} = 0 \quad (2)$$

And the relations of stiffness and mass matrices can be obtained as follows [6]:

$$\boldsymbol{k}_1^{(e)} = \int_{\Omega_e} \left[\beta_1^T \boldsymbol{C}^{(e)} \beta_1\right] d\Omega_e \quad (3)$$

$$\boldsymbol{k}_2^{(e)} = \int_{\Omega_e} \left[\beta_1^T \boldsymbol{C}^{(e)} \beta_2 - \beta_2^T \boldsymbol{C}^{(e)} \beta_1\right] d\Omega_e \quad (4)$$

$$\boldsymbol{k}_3^{(e)} = \int_{\Omega_e} \left[\beta_2^T \boldsymbol{C}^{(e)} \beta_2\right] d\Omega_e \quad (5)$$

$$\boldsymbol{m}^{(e)} = \int_{\Omega_e} \left[\boldsymbol{N}^T \rho^{(e)} \boldsymbol{N}\right] d\Omega_e \quad (6)$$

The relations of all elements can be assembled together to derive the overall relations.

3. Hierarchical shape functions based on the normalized integral of Legendre polynomials:

The main difference between hierarchical shape functions and other shape functions is that in hierarchical shape functions new points are added without moving existing points, and as the order increases, all lower order shape functions are present unchanged in the higher order shape function. Considering the natural coordinate system for the standard one-dimensional element, the basic hierarchical shape functions are as follows:

$$N_1(\xi) = \frac{1}{2}(1-\xi) \quad (7)$$

$$N_2(\xi) = \frac{1}{2}(1+\xi) \quad (8)$$

And the inner shape functions which are composed of the normalized integrals of Legendre polynomials are as follows:

$$N_n^{FEM,p}(\xi) = \frac{1}{\|le_{n-2}\|} \int_{x_1=-1}^{\xi} le_{n-2}(x_1)\, dx_1, \quad (9)$$

$$n = 3,4,5,\dots,p$$

In the above relations, le_{n-2} is a Legendre polynomial of order n-2. It is worth noting that, in this method, the degrees of freedom do not represent the displacement of real nodes [28].

4. Results and Discussion

To investigate the effects of using hierarchical shape functions on the accuracy and computational cost of the SAFE method, h-refinement and p-refinement approaches using high-order shape functions for one isotropic and one anisotropic plate were considered. To facilitate comparison of the results, the results were compared in the same degrees of freedom. The different methods considered here are as follows.

A- h-refinement approach using standard second-order shape functions based on Lagrange polynomials with the same nodal distribution (FEM, h-refinement).

B- p-refinement approach using high order shape functions (p > 2).

In the p-refinement approach, the following shape functions are investigated:

Standard shape functions based on Lagrange polynomials with equal nodal distribution (FEM, p-refinement), spectral shape functions based on Lagrange polynomials with Gauss-Lobatto-Legendre nodal distribution (SEM 1), and Gauss-Lobatto-Chebyshev nodal distribution (SEM 2), and hierarchical shape functions based on the normalized integral of Legendre polynomials (p-FEM).

The percentage of discrepancy between the modeling performed and the results presented using analytical method is calculated using the following relation:

$$\textit{percentage of discrepancy} = \left|\frac{\kappa_{DC} - \kappa_{SAFE}}{\kappa_{DC}}\right| \times 100 \quad (10)$$

Where $\kappa_{DC}$the wavenumber is obtained from "Dispersion Curve Calculator" software and κ_{SAFE} is the wavenumber obtained from the semi-analytical finite element method.

4.1. Case I: Isotropic material

As a first case, a steel plate with a thickness of one millimeter is considered. The mechanical properties of the plate considered in the modeling are listed in Table 1.

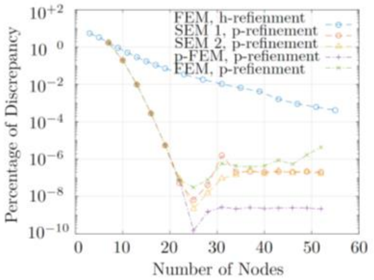
Table 1. Mechanical Properties of Steel plate

G (GPA)	ν	ρ(kg/m^3)
180	0.3	7850

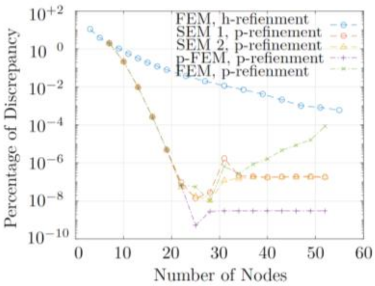
The modeling assumes that the wave propagates along the x-axis, the z-axis is considered along the thickness of the plate, and the y-axis is considered along the width of the plate. It is also assumed that the width of the plate is infinite. For isotropic materials, shear waves involving displacement along the width of the plane can be separated from Lamb waves. Thus, if we wish to consider only Lamb waves, we can ignore the displacements in the y-direction at each node.

The convergence study using the different approaches mentioned and different shape functions is shown in Figure 1. Only the first two modes are used here for comparison.

The first point that can be clearly seen in Figure 1 is that the p-refinement approach with higher order shape functions shows much better performance than the h-refinement approach. The next important point is that hierarchical shape functions perform better than spectral shape functions. The better performance of this method can be verified from two points of view.



a



b

Figure 1. Percentage of discrepancy vs. the number of nodes for an steel plate a) 1st and mode b) 2nd mode

First, we note that the use of higher order hierarchical shape functions has higher accuracy than other shape functions with the same number of degrees of freedom. This means that when considering the specific level of accuracy using high order shape functions, the desired result can be achieved with a lower number of degrees of freedom. Moreover, after converging to a certain value, good stability in the results is observed.

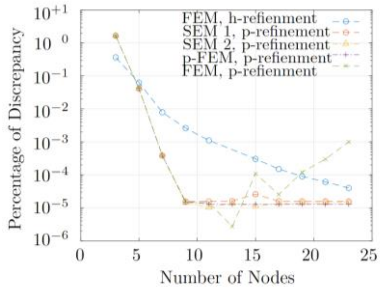
4.2. Case II: Anisotropic material

In this case, a 6 layer composite plate with stacking sequence of $[\pm\mathbf{60},\mathbf{0}]_{\mathbf{s}}$ is considered. The properties of the considered single layer are presented in Table 2.

In this case, the same approach is used for isotropic plates. Figure 2 shows the percentage of discrepancy versus the number of nodes in the layer obtained by the presented methods. As can be seen, the results resemble the case of isotropic material. However, the performance of the different shape functions of the methods used appears to be almost the same. This is mainly due to the increase in degrees of freedom caused by the number of layers.

5. Conclusions

In this article, the effect of using hierarchical shape functions in the SAFE method, as well as its performance has been studied. The simulation results presented here show that the high-order shape functions commonly used in the spectral finite element method and the p-version of finite element method exhibit very good performance in improving the efficiency and reducing the computational cost of this method. Moreover, according to the research and comparison between high-order shape functions, it can be stated that among the considered methods, hierarchical shape functions are more accurate considering the same number of degrees of freedom.



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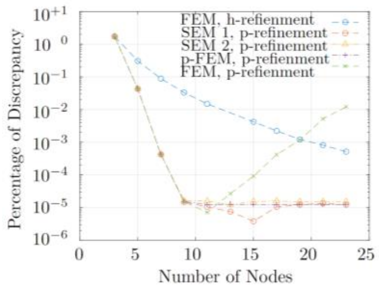

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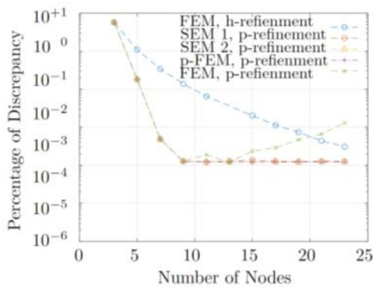

c

Figure 2. Percentage of discrepancy vs. the number of nodes in each layer for composite plate with stacking sequence of $[\pm\mathbf{60},\mathbf{0}]_{\mathbf{s}}$ a) 1st mode, b) 2nd mode and c) 3rd mode

Table 2. Mechanical properties of a single layer AS4/3502

E_1(GPa)	E_2(GPa)	E_3(GPa)	G_{12}(GPa)	G_{13}(GPa)	G_{23}(GPa)	ν_{13}	ν_{23}	ρ (kg/m^3)
127.6	11.3	11.3	5.97	3.75	3.75	0.3	0.3	1578

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